FIELDS OF PARALLEL VECTORS IN A RIEMANNIAN GEOMETRY*

BY

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1. Introduction. Consider a space V_n of n dimensions whose fundamental quadratic form is

$$(1.1) \qquad \varphi = g_{ij}\, dx^i\, dx^j \qquad\qquad (i, j = 1, \cdots, n),$$

where the g's are subject only to the condition that their determinant

$$(1.2) \qquad g = |g_{ij}|$$

is not zero.† We do not impose the condition that φ is definite. We define the metric by

$$(1.3) \qquad ds^2 = e g_{ij}\, dx^i\, dx^j,$$

where e is plus or minus one according as φ is positive or negative for a given set of values of the differentials.

Consider any curve C in V_n which is not of length zero, that is for which φ is not zero, and assume that the coördinates x^i at points of C are functions of s measured from some point of C along it. Let λ^i be the contravariant components of a unit vector field in V_n, that is

$$g_{ij}\, \lambda^i \lambda^j = e,$$

where e is plus or minus one. In accordance with the concept of parallelism in V_n due to Levi-Civita, we say that the vectors of the field at points of C are parallel to one another with respect to C, if

$$(1.4) \qquad \frac{dx^k}{ds}\, \lambda^i{}_{,k} \equiv \frac{dx^k}{ds}\left(\frac{\partial \lambda^i}{\partial x^k} + \left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\} \lambda^j \right) = 0,$$

where $\lambda^i{}_{,k}$ is the covariant derivative of λ^i, $\left\{ \begin{matrix} i \\ j\,k \end{matrix} \right\}$ being the Christoffel symbols of the second kind formed with respect to (1.1). When

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† We use the customary convention of summation indicated by a repeated index.

(1.5) $$\lambda^i{}_{,k} = 0,$$

and only in this case, any two vectors are parallel to one another with respect to any curve passing through their points. In this case we have by definition a *field of parallel vectors*.

Fields of parallel vectors for geometries of paths have been considered by the author* and by Veblen and Thomas;† the results of the latter can be used to give in invariantive form the conditions that a Riemannian space shall admit one or more fields of parallel vectors. In the present paper we derive the canonical forms of (1.1) for such spaces.‡

2. **Canonical coördinate system.** Suppose that a V_n is such that equations (1.5) admit p linearly independent sets of solutions, $\lambda_{\sigma|}{}^i$ for $\sigma = 1, \cdots, p$ and $i = 1, \cdots, n.$§ In any other coördinate system, x'^i, the components $\lambda'_{\sigma|}{}^i$ are given by

(2.1) $$\lambda'_{\sigma|}{}^i = \lambda_{\sigma|}{}^j \frac{\partial x'^i}{\partial x^j}.$$

If we take the system of p differential equations

(2.2) $$X_\sigma(\theta) \equiv \lambda_{\sigma|}{}^j \frac{\partial \theta}{\partial x^j} = 0$$

and apply the Poisson operator, we have in consequence of (1.5)

$$(X_\sigma X_\tau - X_\tau X_\sigma)\, \theta \equiv 0.$$

Therefore equations (2.2) form a complete system, and consequently admit $n - p$ independent solutions.|| If we take them for the coördinates $x'^{p+1}, \cdots, x'^n$, it follows from (2.1) that $\lambda'_{\sigma|}{}^t = 0$ for $t = p+1, \cdots, n$. Again if we omit one of the equations from (2.2), say $X_r\theta = 0$, the remaining system is complete and admits in addition to $x'^{p+1}, \cdots, x'^n$ another independent solution x'^r. In this way the x's are defined so that all of the components of the λ's are zero except those with the same

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*Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 207–212.

† These Transactions, vol. 25 (1923), p. 589.

‡ The case of spaces admitting one field of parallel vectors was considered by Levi-Civita, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173–205.

§ Unless otherwise stated, it will be assumed that Greek indices take the values $1, \cdots, p$ and Latin $1, \cdots, n$, throughout this paper.

|| Goursat, *Leçons sur l'Intégration des Equations aux Dérivées Partielles du Premier Ordre*, p. 52.

subscript and superscript. If it is assumed that these vectors are unit vectors, we have in the new coördinate system

$$(2.3) \quad \lambda_{\sigma|}{}^{\sigma} = \frac{1}{\sqrt{e_\sigma\, g_{\sigma\sigma}}}, \quad \lambda_{\sigma|}{}^{t} = 0 \quad (\sigma = 1, \cdots, p;\ t = 1, \cdots, n;\ t \neq \sigma).$$

When these expressions are substituted in (1.5) we get

$$\frac{\partial}{\partial x^k} \log \sqrt{g_{\sigma\sigma}} - \begin{Bmatrix} \sigma \\ k\ \sigma \end{Bmatrix} = 0 \qquad\qquad (\sigma \text{ not summed}),$$

$$\begin{Bmatrix} j \\ k\ \sigma \end{Bmatrix} = 0 \quad (\sigma = 1, \cdots, p;\ j, k = 1, \cdots, n;\ j \neq \sigma).$$

If we multiply the first of these equations by $g_{\sigma l}$ and subtract the second multiplied by g_{jl} and summed for j, we get the equivalent set of equations

$$g_{\sigma l}\, \frac{\partial}{\partial x^k} \log \sqrt{g_{\sigma\sigma}} - [k\,\sigma,\, l] = 0,$$

that is

$$(2.4) \quad g_{\sigma l}\, \frac{\partial}{\partial x^k} \log g_{\sigma\sigma} - \frac{\partial g_{kl}}{\partial x^\sigma} - \frac{\partial g_{\sigma l}}{\partial x^k} + \frac{\partial g_{k\sigma}}{\partial x^l} = 0.$$

For the case $k = \sigma$, these equations reduce to

$$\frac{\partial}{\partial x^\sigma} \left(\frac{g_{\sigma l}}{\sqrt{e_\sigma\, g_{\sigma\sigma}}} \right) = e_\sigma\, \frac{\partial}{\partial x^l} \sqrt{e_\sigma\, g_{\sigma\sigma}}.$$

In accordance with these equations we define p functions ψ_σ by

$$\sqrt{e_\sigma\, g_{\sigma\sigma}} = e_\sigma\, \frac{\partial \psi_\sigma}{\partial x^\sigma}, \qquad \frac{g_{\sigma l}}{\sqrt{e_\sigma\, g_{\sigma\sigma}}} = \frac{\partial \psi_\sigma}{\partial x^l},$$

from which we have

$$(2.5) \quad g_{\sigma l} = e_\sigma\, \frac{\partial \psi_\sigma}{\partial x^\sigma}\, \frac{\partial \psi_\sigma}{\partial x^l}.$$

From these expressions it follows that ψ_σ must involve x^σ, otherwise the space is of less than n dimensions.

Again if neither k nor l in (2.4) is σ, we have

$$(2.6) \quad g_{kl} = e_\sigma\, \frac{\partial \psi_\sigma}{\partial x^k}\, \frac{\partial \psi_\sigma}{\partial x^l} + \varphi_{kl\sigma} \quad \begin{pmatrix} \sigma = 1, \cdots, p;\ k, l = 1, \cdots, n; \\ k \neq \sigma,\ l \neq \sigma \end{pmatrix},$$

where $\varphi_{kl\sigma}$ is a function independent of x^σ.

From (2.5) and (2.6) it follows that for each value of σ the fundamental form can be written

$$\varphi = e_\sigma (d\psi_\sigma)^2 + g_{rs}\, dx^r\, dx^s \qquad (r, s = 1, \cdots, n;\; r \neq \sigma,\, s \neq \sigma),$$

where g_{rs} are independent of x^σ. If then we put $x'^\sigma = \psi^\sigma$, $x'^j = x^j\; (j \neq \sigma)$, the curves of parameter x'^σ are the same as those of parameter x^σ, and these curves are geodesics.* Hence we have:

When a V_n admits p independent fields of parallel vectors, the vectors of each field are the tangent vectors to a congruence of geodesics.

Conversely if the fundamental form of any space is reducible to the form

(2.7) $$\varphi = e_1\, dx_1^2 + g_{rs}\, dx^r\, dx^s \qquad (r, s = 2, \cdots, n),$$

it is found from (2.4) that a necessary and sufficient condition that the tangents to the curves of parameter x^1 form a parallel field is that g_{rs} be independent of x^1. In this case all the spaces $x^1 = $ const. have the same fundamental form and consequently any one of them can be brought into coincidence with any other by a *translation*, that is by a motion in which each point describes the same distance along the geodesic normal to the sub-space. In the case $p > 1$ the space admits p independent translations; thus any two of the sub-spaces of each of the family of sub-spaces $\psi_\sigma = $ const. can be brought into coincidence by a translation.

If in particular we take $\psi_\sigma = x^\sigma + \varphi_\sigma(x^{p+1}, \cdots, x^n)$ for $\sigma = 1, \cdots, p$, it follows from (2.5) and (2.6) that for a V_n with the fundamental form

(2.8) $$\varphi = e_1\,(dx^1)^2 + \cdots + e_p\,(dx^p)^2 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = p+1, \cdots, n),$$

where $g_{\alpha\beta}$ are arbitrary functions of $x^{p+1}, \cdots, x^n$, the tangents to curves of parameters $x^1, \cdots, x^p$ form p fields of parallel vectors. We proceed to the determination of the general solution.

3. **When $p > 1$.** When $p > 1$, we have from (2.5), on equating the two expressions for $g_{\sigma\tau}$,

(3.1) $$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\sigma}{\partial x^\tau} = e_\tau \frac{\partial \psi_\tau}{\partial x^\sigma} \frac{\partial \psi_\tau}{\partial x^\tau},$$

and from (2.5) and (2.6), on equating the two expressions for $g_{\tau\tau}$ and $g_{\sigma\sigma}$ respectively,

(3.2)
$$e_\sigma \left(\frac{\partial \psi_\sigma}{\partial x^\tau} \right)^2 + g_{\tau\tau\sigma} = e_\tau \left(\frac{\partial \psi_\tau}{\partial x^\tau} \right)^2,$$
$$e_\tau \left(\frac{\partial \psi_\tau}{\partial x^\sigma} \right)^2 + g_{\sigma\sigma\tau} = e_\sigma \left(\frac{\partial \psi_\sigma}{\partial x^\sigma} \right)^2.$$

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* Bianchi, *Lezioni*, vol. 1, p. 337.

When these equations are differentiated with respect to x^σ and x^τ respectively, we obtain

$$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\tau} \frac{\partial^2 \psi_\sigma}{\partial x^\sigma \partial x^\tau} - e_\tau \frac{\partial \psi_\tau}{\partial x^\tau} \frac{\partial^2 \psi_\tau}{\partial x^\sigma \partial x^\tau} = 0,$$

$$e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial^2 \psi_\sigma}{\partial x^\sigma \partial x^\tau} - e_\tau \frac{\partial \psi_\tau}{\partial x^\sigma} \frac{\partial^2 \psi_\tau}{\partial x^\sigma \partial x^\tau} = 0.$$

Hence either

$$(3.3) \qquad \frac{\partial^2 \psi_\sigma}{\partial x^\sigma \partial x^\tau} = \frac{\partial^2 \psi_\tau}{\partial x^\sigma \partial x^\tau} = 0,$$

or

$$(3.4) \qquad \frac{\partial \psi_\sigma}{\partial x^\tau} \frac{\partial \psi_\tau}{\partial x^\sigma} = \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\tau}{\partial x^\tau}.$$

If (3.4) holds, it follows from (3.1) and (3.2) that $\varphi_{\sigma\sigma\tau} = \varphi_{\tau\tau\sigma} = 0$. This case will be considered in § 7.

From equations of the form (3.3) it follows that the functions ψ_σ must be of the form

$$(3.5) \quad \begin{aligned} \psi_\sigma &= e_\sigma f_\sigma \left(x^\sigma, x^{p+1}, \cdots, x^n \right) \\ &+ F_\sigma (x^1, \cdots, x^{\sigma-1}, x^{\sigma+1}, \cdots, x^p, x^{p+1}, \cdots, x^n) \quad (\sigma = 1, \cdots, p). \end{aligned}$$

From the remark following (2.5) it is seen that f_σ must involve x^σ. Moreover it is understood that f_σ does not have any additional function independent of x^σ, since all such terms are included in F_σ.

If $\partial \psi_\sigma / \partial x^\tau = 0$, so also is $\partial \psi_\tau / \partial x^\sigma = 0$, as follows from (3.1). These exceptional cases are taken care of by (3.5).

Substituting from (3.5) in (3.1), we get

$$(3.6) \qquad \frac{\dfrac{\partial F_\tau}{\partial x^\sigma}}{\dfrac{\partial f_\sigma}{\partial x^\sigma}} = \frac{\dfrac{\partial F_\sigma}{\partial x^\tau}}{\dfrac{\partial f_\tau}{\partial x^\tau}}.$$

Since the first and second members of this equation are independent of x^τ and x^σ respectively, they are independent of both. Consequently we replace (3.6) by

$$(3.7) \quad \frac{\partial F_\tau}{\partial x^\sigma} = \frac{\partial f_\sigma}{\partial x^\sigma} \omega_{\tau\sigma}, \quad \frac{\partial F_\sigma}{\partial x^\tau} = \frac{\partial f_\tau}{\partial x^\tau} \omega_{\sigma\tau} \quad (\sigma, \tau = 1, \cdots, p; \ \sigma \neq \tau),$$

where $\omega_{\tau\sigma} (= \omega_{\sigma\tau})$ is independent of x^σ and x^τ.

4. When $p = 2$. When $p = 2$, we have from (3.5) and (3.7), since $\omega_{\sigma\tau}$ is symmetric,

$$(4.1) \qquad \psi_\sigma = e_\sigma f_\sigma + A f_\tau + B_\sigma \qquad (\sigma, \tau = 1, 2; \sigma \neq \tau),$$

where A, B_1 and B_2 are independent of x^1 and x^2.

For the general case we have from (2.5) and the expressions for $g_{\sigma l}$ from (2.6) the equations of condition

$$(4.2) \quad e_\sigma \frac{\partial \psi_\sigma}{\partial x^\sigma} \frac{\partial \psi_\sigma}{\partial x^l} = e_\tau \frac{\partial \psi_\tau}{\partial x^\sigma} \frac{\partial \psi_\tau}{\partial x^l} + \varphi_{\sigma l \tau} \quad \begin{pmatrix} \sigma = 1, \cdots, p; \, l = 1, \cdots, n; \\ l, \, \sigma, \, \tau \neq \end{pmatrix}.$$

When $p = 2$ and we take $\sigma = 1$, $\tau = 2$, $l = 3, \cdots, n$, this condition for the expressions (4.1) becomes

$$\frac{\partial f_1}{\partial x^1} \left[f_2 \frac{\partial A}{\partial x^l} + \frac{\partial}{\partial x^l} (e_1 f_1 + B_1) - e_2 A_2 \frac{\partial}{\partial x^l} (A f_1 + B_2) \right] - \varphi_{1\sigma 2} = 0.$$

Since $\partial f_1 / \partial x^1 \neq 0$ and all the terms except f_2 are independent of x^2, it follows that A must be a constant.

There remain for consideration the equations which arise from (2.6) on equating the expressions for each g_{kl} $(k, l = 3, \cdots, n)$ for $\sigma = 1, 2$. Since φ_{kl1} and φ_{kl2} may be any functions not involving x^1 and x^2 respectively, we find that these conditions are satisfied, because A is a constant. In arriving at this result we take

$$\varphi_{kl1} = e_2 \frac{\partial}{\partial x^k} (e_2 f_2 + B_2) \frac{\partial}{\partial x^l} (e_2 f_2 + B_2) - e_1 \frac{\partial}{\partial x^k} (A f_2 + B_1) \frac{\partial}{\partial x^l} (A f_2 + B_1)$$
$$(4.3)$$
$$+ e_1 \frac{\partial B_1}{\partial x^k} \frac{\partial B_1}{\partial x^l} + C_{kl},$$

where C_{kl} are arbitrary functions of $x^3, \cdots, x^n$. Then we have

$$(4.4) \qquad g_{kl} = \sum_{\sigma}^{1,2} e_\sigma \frac{\partial}{\partial x^k} (e_\sigma f_\sigma + B_\sigma) \frac{\partial}{\partial x^l} (e_\sigma f_\sigma + B_\sigma)$$
$$+ A \left(\frac{\partial f_1}{\partial x^k} \frac{\partial f_2}{\partial x^l} + \frac{\partial f_1}{\partial x^l} \frac{\partial f_2}{\partial x^k} \right) + C_{kl}.$$

Hence we have the following theorem:

When f_1 and f_2 are arbitrary functions of $x^1, x^3, \cdots, x^n$ and $x^2, x^3, \cdots, x^n$ respectively, A is an arbitrary constant, and B_1 and B_2 are arbitrary func-

tions of $x^3, \cdots, x^n$, *the functions* (4.1) *determine a* V_n *with two fields of parallel vectors; and* C_{kl} *in* (4.4) *are also arbitrary functions of* $x^3, \cdots, x^n$.

5. **When** $p = 3$. If $p > 2$ and we express the condition of integrability of two of the first set of equations (3.7) for two values of σ different from τ, say α and β, we find by considerations similar to those applied to (3.6)

$$\frac{\partial \omega_{\tau\alpha}}{\partial x^\beta} = \frac{\partial f_\beta}{\partial x^\beta} \omega_{\tau\alpha\beta}, \qquad \frac{\partial \omega_{\tau\beta}}{\partial x^\alpha} = \frac{\partial f_\alpha}{\partial x^\alpha} \omega_{\tau\beta\alpha} \qquad \left(\begin{matrix} \alpha,\ \beta,\ \tau = 1, \cdots, p; \\ \alpha,\ \beta,\ \tau \ \mp \end{matrix} \right),$$

where $\omega_{\tau\alpha\beta} (= \omega_{\tau\beta\alpha})$ is independent of x^τ, x^α and x^β, and consequently

$$(5.1) \qquad\qquad\qquad \omega_{\tau\alpha} = f_\beta\, \omega_{\tau\alpha\beta} + \varrho_{\tau\alpha\beta},$$

where $\varrho_{\tau\alpha\beta}$ is independent of x^τ, x^α and x^β. Since $\omega_{\tau\alpha} = \omega_{\alpha\tau}$, $\omega_{\tau\alpha\beta}$ must be symmetric in τ, α and β, and $\varrho_{\tau\alpha\beta}$ in τ and α. For $p = 3$, we have from (3.7) and (5.1)

$$(5.2) \qquad\qquad F_\sigma = A \prod_\alpha{}'_\sigma (f_\alpha) + \sum_\alpha{}'_\sigma B_{\sigma\alpha} f_\alpha + C_\sigma \qquad (\sigma = 1, 2, 3),$$

where $\prod_\alpha{}'_\sigma (f_\alpha)$ means the product of all the f's except f_σ and $\sum_\alpha{}'_\sigma$ indicates the sum for α except $\alpha = \sigma$; also A, $B_{\sigma\alpha} = B_{\alpha\sigma}$ and C_σ are functions independent of x^1, x^2 and x^3.

If we substitute in (4.2) the expressions $\psi_\sigma = e_\sigma f_\sigma + F_\sigma$, where F_σ is given by (5.2) and similarly for ψ_τ, we obtain equations of the form

$$\frac{\partial f_\sigma}{\partial x^\sigma} f_\tau \frac{\partial}{\partial x^l} (f_\varrho A + B_{\sigma\tau}) + \Phi_\tau = 0 \qquad \left(\begin{matrix} \varrho,\ \sigma,\ \tau = 1, 2, 3; \quad \varrho,\ \sigma,\ \tau \ \mp \\ l = 1, \cdots, n; \ l,\ \sigma,\ \tau \ \mp \end{matrix} \right),$$

where Φ_τ does not involve x^τ. Hence

$$\frac{\partial}{\partial x^l} (f_\varrho A + B_{\sigma\tau}) = 0.$$

For $l = \varrho$ we get $A = 0$, and for other values of l it follows that the B's are constant.

When as in § 4 we equate the three values of each g_{kl} ($k, l = 4, \cdots, n$) obtained from (2.6) by taking $\sigma = 1, 2, 3$, we find that these equations are consistent in view of the above results. Moreover, the functions $\varphi_{kl\sigma}$ are of the forms

$$\varphi_{kl\sigma} = e_\tau \frac{\partial}{\partial x^k}(e_\tau f_\tau + B_{\tau\varrho}f_\varrho + C_\tau)\frac{\partial}{\partial x^l}(e_\tau f_\tau + B_{\tau\varrho}f_\varrho + C_\tau)$$

$$- e_\alpha B_{\sigma\tau}^2 \frac{\partial f_\tau}{\partial x^k}\frac{\partial f_\tau}{\partial x^l} + D_{kl\sigma\tau}$$

$$(5.3) \qquad = e_\varrho \frac{\partial}{\partial x^k}(e_\varrho f_\varrho + B_{\tau\varrho}f_\tau + C_\varrho)\frac{\partial}{\partial x^l}(e_\varrho f_\varrho + B_{\tau\varrho}f_\tau + C_\varrho)$$

$$- e_\alpha B_{\sigma\varrho}^2 \frac{\partial f_\varrho}{\partial x^k}\frac{\partial f_\varrho}{\partial x^l} + D_{kl\sigma\varrho}$$

$$(\varrho,\,\sigma,\,\tau = 1,\,2,\,3;\quad \varrho,\,\sigma,\,\tau \neq),$$

where $D_{kl\sigma\tau}$ $(= D_{kl\tau\sigma})$ is independent of x^σ and x^τ. These two expressions for $\varphi_{kl\sigma}$ are seen to be consistent, if we take

$$D_{kl\sigma\tau} = e_\varrho \frac{\partial f_\varrho}{\partial x^k}\frac{\partial f_\varrho}{\partial x^l} + \frac{\partial f_\varrho}{\partial x^k}\frac{\partial C_\varrho}{\partial x^l} + \frac{\partial f_\varrho}{\partial x^l}\frac{\partial C_\varrho}{\partial x^k}$$

$$(5.4) \qquad\quad - e_\tau \frac{\partial}{\partial x^k}(B_{\tau\varrho}f_\varrho + C_\tau)\frac{\partial}{\partial x^l}(B_{\tau\varrho}f_\varrho + C_\tau)$$

$$- e_\sigma \frac{\partial}{\partial x^k}(B_{\sigma\varrho}f_\varrho + C_\sigma)\frac{\partial}{\partial x^l}(B_{\sigma\varrho}f_\varrho + C_\sigma) + E_{kl\sigma\varrho},$$

where $E_{kl\sigma\varrho}$ are arbitrary functions of $x^4, \cdots, x^a$. Hence we have the following theorem:

The functions ψ_σ defined by

$$\psi_\sigma = e_\sigma f_\sigma + \sum_\alpha{}' B_{\sigma\alpha}f_\alpha + C_\sigma,$$

where the f_σ are arbitrary functions of $x^\sigma, x^4, \cdots, x^n$, the B's are constants symmetric in their indices and the C's are arbitrary functions of $x^4, \cdots, x^n$, determine a V_n with three fields of parallel vectors; and $E_{kl\sigma\varrho}$ in (5.4) are arbitrary functions of $x^4, \cdots, x^n$.

6. **When** $p > 3$. In this case we have from (5.1) equations of the form

$$(6.1) \qquad\qquad f_\beta\,\omega_{\tau\alpha\beta} + \varrho_{\tau\alpha\beta} = f_\gamma\,\omega_{\tau\alpha\gamma} + \varrho_{\tau\alpha\gamma} \qquad\qquad (\tau,\,\alpha,\,\beta,\,\gamma \neq).$$

If this equation be differentiated with respect to x^β and x^γ, the resulting equation leads, by considerations similar to those applied to (3.6), to

$$\frac{\partial\omega_{\tau\alpha\beta}}{\partial x^\gamma} = \frac{\partial f_\gamma}{\partial x^\gamma}\,\omega_{\tau\alpha\beta\gamma},\qquad \frac{\partial\omega_{\tau\alpha\gamma}}{\partial x^\beta} = \frac{\partial f_\beta}{\partial x^\beta}\,\omega_{\tau\alpha\gamma\beta},$$

where $\omega_{\tau\alpha\beta\gamma}$ is symmetric in β and γ and is independent of x^τ, x^α, x^β and x^γ. From these equations we have

$$\omega_{\tau\alpha\beta} = f_\gamma\,\omega_{\tau\alpha\beta\gamma} + \varrho_{\tau\alpha\beta\gamma}, \qquad \omega_{\tau\alpha\gamma} = f_\beta\,\omega_{\tau\alpha\gamma\beta} + \varrho_{\tau\alpha\gamma\beta},$$

and from (6.1)

$$\varrho_{\tau\alpha\beta} = \varrho_{\tau\alpha\gamma\beta}\,f_\gamma + \bar\varrho_{\tau\alpha\beta\gamma}, \qquad \varrho_{\tau\alpha\gamma} = \varrho_{\tau\alpha\beta\gamma}\,f_\beta + \bar\varrho_{\tau\alpha\gamma\beta}.$$

By continuing this process, we find as the general solution of (3.7)

$$
(6.2)\quad
\begin{aligned}
F_\sigma = {} & A_\sigma \prod_\alpha{}'_\sigma (f_\alpha) + \sum_{\varrho_1}{}'_\sigma A_{\sigma\varrho_1} \prod_\alpha{}'_{\sigma\varrho_1}(f_\alpha) + \sum_{\varrho_1,\varrho_2}{}'_\sigma A_{\sigma\varrho_1\varrho_2} \prod_\alpha{}'_{\sigma\varrho_1\varrho_2}(f_\alpha) + \cdots \\
& + \sum_{\varrho_1,\cdots,\varrho_{p-2}}{}'_\sigma A_{\sigma\varrho_1\cdots\varrho_{p-2}} \prod_\alpha{}'_{\sigma\varrho_1\cdots\varrho_{p-2}}(f_\alpha) + C_\sigma,
\end{aligned}
$$

where the A's and C's are functions of $x^{p+1}, \cdots, x^n$, the symbol $\sum'_\sigma{}_{\varrho_1\cdots\varrho_r}$ means the sum for all different combinations of different ϱ's taking the values $1, \cdots, p$ except σ and only one such term for each combination; and $\prod_\alpha{}'_{\sigma\varrho_1\cdots\varrho_r}$ means the product of the f's except $f_\sigma, f_{\varrho_1}, \cdots, f_{\varrho_r}$. In order that (3.1) be satisfied, we must have $A_1 = A_2 = \cdots = A_p \equiv A$ and the other A's must satisfy the conditions

$$(6.3)\qquad A_{\sigma\varrho_1\cdots\varrho_r} = A_{\tau\varrho_1\cdots\varrho_r} \qquad (\sigma, \tau, \varrho_1, \cdots, \varrho_r \neq).$$

When these expressions are substituted in (4.2), the latter are reducible to the form $\Phi_\sigma f_\tau + \Psi_\sigma = 0$ where Φ_σ and Ψ_σ are independent of x^τ. Accordingly they must vanish. The first of these conditions is

$$
(6.4)\quad
\begin{aligned}
\frac{\partial}{\partial x^l} \Big[& A \prod{}'_{\sigma\tau}(f_\alpha) + \sum_{\varrho_1}{}'_{\sigma\tau} A_{\sigma\varrho_1} \prod{}'_{\sigma\tau\varrho_1}(f_\alpha) \\
& + \sum_{\varrho_1,\varrho_2}{}'_{\sigma\tau} A_{\sigma\varrho_1\varrho_2} \prod{}'_{\sigma\tau\varrho_1\varrho_2}(f_\alpha) + \cdots + D_{\sigma\tau} \Big] = 0 \quad (l, \sigma, \tau \neq),
\end{aligned}
$$

where

$$(6.5)\qquad D_{\sigma\tau} = A_{\sigma\varrho_1\cdots\varrho_{p-2}} = A_{\tau\varrho_1\cdots\varrho_{p-2}} \qquad (\varrho_1, \cdots, \varrho_{p-2}, \sigma, \tau \neq).$$

When l takes a value $1, \cdots, p$, different from σ and τ, say ϱ, the above equation becomes

$$
\frac{\partial f_\varrho}{\partial x^\varrho} \Big[A \prod{}'_{\varrho\sigma\tau}(f_\alpha) + \sum_{\varrho_1}{}'_{\varrho\sigma\tau} A_{\sigma\varrho_1} \prod{}'_{\varrho\sigma\tau\varrho_1}(f_\alpha) \\
+ \sum_{\varrho_1,\varrho_2}{}'_{\varrho\sigma\tau} A_{\sigma\varrho_1\varrho_2} \prod{}'_{\varrho\sigma\tau\varrho_1\varrho_2}(f_\alpha) + \cdots + E_{\varrho\sigma\tau} \Big] = 0,
$$

where

$$E_{\varrho\sigma\tau} = A_{\sigma\varrho_1\cdots\varrho_{p-2}} = A_{\tau\varrho_1\cdots\varrho_{p-2}} = A_{\varrho\varrho_1\cdots\varrho_{p-2}} \quad (\varrho_1, \cdots, \varrho_{p-3}, \varrho, \sigma, \tau \neq).$$

Since $\partial f_\varrho/\partial x^\varrho \neq 0$, it follows from the above equations that A and all the A's with less than $p-1$ indices are zero. And from (6.4) for $l = p+1, \cdots, n$ we have that the D's (6.5) are constants. Hence from (3.5) and (6.2) we have

$$(6.6) \qquad \psi_\sigma = e_\sigma f_\sigma + \sum_\tau{}'_\sigma D_{\sigma\tau} f_\tau + C_\sigma \quad (\sigma, \tau = 1, \cdots, p; \, \sigma \neq \tau),$$

where the D's are constants symmetric in the subscripts and C_σ are arbitrary functions of $x^{p+1}, \cdots, x^n$.

When now we equate the p expressions for each of the functions $g_{kl} (k, l = p+1, \cdots, n)$ obtained from (2.6) by taking $\sigma = 1, \cdots, p$, we find that these equations are consistent and that the functions $\varphi_{kl\sigma}$ involve additive arbitrary functions of $x^{p+1}, \cdots, x^n$ as shown in §§ 4, 5. Thus the theorems of these sections can be generalized to any value of $p < n$.

7. **When one or more of the functions** $\varphi_{\sigma\sigma\tau}$ **in (3.2) are zero.** If $\varphi_{\sigma\sigma\tau} = 0$ in the second of (3.2), it follows from this equation that $e_\sigma = e_\tau$, if the functions are to be real. From this equation and (3.1) we have then

$$(7.1) \qquad \left(\frac{\partial \psi_\tau}{\partial x^\sigma}\right)^2 = \left(\frac{\partial \psi_\sigma}{\partial x^\sigma}\right)^2, \qquad \left(\frac{\partial \psi_\tau}{\partial x^\tau}\right)^2 = \left(\frac{\partial \psi_\sigma}{\partial x^\tau}\right)^2,$$

and from the first of (3.2) we find that $\varphi_{\tau\tau\sigma} = 0$.

If we take $\sigma = 1$, $\tau = 2$, from the first of (7.1) it follows that

$$(7.2) \qquad \psi_2 = \varepsilon \psi_1 + \varphi_1,$$

where $\varepsilon^2 = 1$ and φ_1 is independent of x^1. Substituting in the second of (7.1), we obtain

$$\left(2\varepsilon \frac{\partial \psi_1}{\partial x^2} + \frac{\partial \varphi_1}{\partial x^2}\right) \frac{\partial \varphi_1}{\partial x^2} = 0.$$

Assume that $\partial \varphi_1/\partial x^2 \neq 0$; then

$$(7.3) \qquad \psi_1 = \frac{1}{2\varepsilon} (\varphi_2 - \varphi_1),$$

where φ_2 is independent of x^2, and from (7.2)

$$\psi_2 = \frac{1}{2} (\varphi_2 + \varphi_1).$$

Substituting in (3.1), we get $\partial \varphi_2 / \partial x^1 = 0$. In this case ψ_1 is independent of x^1 and from (2.5) we get $g_{1l} = 0$ $(l = 1, \cdots, n)$, which is impossible. Hence φ_1 must be independent of x^2, and accordingly we write (7.2) in the form

(7.4) $$\psi_2 = \varepsilon \psi_1 + \varphi_{12},$$

where φ_{12} is independent of x^1 and x^2.

When the expression (7.4) is substituted in (4.2) for $\sigma = 1$, $\iota = 2$, we get

$$\varepsilon \frac{\partial \psi_1}{\partial x^1} \frac{\partial \varphi_{12}}{\partial x^l} + \varphi_{1l2} = 0 \qquad (l = 3, \cdots, n).$$

Differentiating with respect to $\dot{x}^2$, we obtain

$$\frac{\partial^2 \psi_1}{\partial x^1 \partial x^2} \frac{\partial \varphi_{12}}{\partial x^l} = 0.$$

Hence either

(7.5) $$\psi_1 = f_1 (x^1, x^3, \cdots, x^n) + f_2' (x^2, x^3, \cdots, x^n),$$

or φ_{12} is a constant. In the latter case we have from (2.5) and (7.4)

$$g_{1l} = e_1 \frac{\partial \psi_1}{\partial x^1} \frac{\partial \psi_1}{\partial x^l}, \qquad g_{2l} = e_1 \frac{\partial \psi_1}{\partial x^2} \frac{\partial \psi_1}{\partial x^l} \qquad (l = 1, \cdots, n).$$

Then the fundamental form may be written

$$\varphi = e_1 (d \psi^1)^2 + g_{\alpha\beta} \, dx^\alpha \, dx^\beta \qquad (\alpha, \beta = 3, \cdots, n),$$

that is, the space is a V_{n-1} at most. Hence (7.5) is the only solution.

It is readily found that the two expressions for any g_{kl} $(k, l, = 3, \cdots, n)$ and for $\sigma = 1, 2$ from (2.6) are consistent under the above conditions.

From the foregoing considerations it follows that the particular cases when one or more of the functions $\varphi_{\sigma\sigma\tau}$ in (3.2) are zero arise only when constants $D_{\sigma\tau}$ in (6.6) have certain values.

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